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Exact Results and Self-Averaging Properties for Random–Random Walks on a One-Dimensional Infinite Lattice

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We present new exact results for a one-dimensional asymmetric disordered hopping model. The lattice is taken infinite from the start and we do not resort to the periodization scheme used by Derrida. An explicit resummation allows for the calculation of the velocity V and the diffusion constant D (which are found to coincide with those given by Derrida) and for demonstrating that V is indeed a self-averaging quantity; the same property is established for D in the limiting case of a directed walk.

KEY WORDS: Fluctuation phenomena; random processes and Brownian motion.

The purpose of this paper is to present new exact results for the dynamics of a particle on an infinite disordered lattice, governed by the following master equation:

$$dp_n/dt = W_{n,n-1}\, p_{n-1}(t) + W_{n,n+1}\, p_{n+1}(t) - (W_{n+1,n} + W_{n-1,n})\, p_n(t) \qquad (1)$$

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giving the probability $p_n(t)$ for the particle to be at site labeled n at time t. The hopping rates $W_{n,n'}$ are positive random quantities possessing bounded inverse moments and the pairs $(W_{n,n+1}, W_{n+1,n})$ are assumed to be independent from one link to the other. In addition, they are taken as asymmetric $(W_{n,n'} \neq W_{n',n})$ as a consequence of a local or external bias field; for definiteness, we shall assume below that the average over disorder of the quantity $\log(W_{n,n+1}/W_{n+1,n})$ is *negative*; the motion to the right is thus favored as compared to the opposite direction. The main two results of this paper are the following.

(i) For a given infinite sample, the velocity V is found to be equal with probability one to the result derived by Derrida[1] for a periodized lattice in the limit of infinite period. Thus, V displays no sample-to-sample fluctuations, i.e., is a self-averaging quantity.

(ii) The *disorder-averaged* diffusion constant D is also found to coincide with the result of ref. 1. In addition, we have shown in the limiting case of a directed walk that D also is self-averaging. This latter result is likely to be true in the general case.

The problem (1) has been solved by Derrida[1] in the case of a periodized sample; he there defines a (large) cell containing N sites with $2N$ random hopping rates, which is repeated at infinity and thus generates an infinite lattice $(W_{n+N,m+N} = W_{n,m})$. Derrida is able to calculate the velocity and diffusion constant for such a periodic sample; taking eventually the limit $N \to \infty$, he finds that these quantities no longer fluctuate. This characterizes a dynamical regime valid once the particle has covered a distance at least equal to one cell (this point is best realized by noting that, for any given N, both V and D depend on *all* the W's defined within such a cell). However, since the particle, in the course of its motion, periodically encounters the *same* sampling for the W's, the question arises whether this limiting procedure actually yields the proper transport coefficients for a lattice taken infinite from the start (that is, whether the limits $N \to +\infty$ and $t \to +\infty$ do commute[1]). In a *disordered* system, this should not be considered as a trivial irrelevance of boundary conditions; indeed, one might think that since the periodized sample is in some sense "semidisordered," fluctuations in the true infinite disordered lattice could be in some way underestimated. Basically, the true question at hand here is that of sample-to-sample fluctuations. To the best of our understanding, even mathematical works on this subject do not settle this question.[2] Moreover, since in the limiting procedure $N \to +\infty$ the position of the particle must simultaneously be pushed at infinity, the physical content of the Derrida regime, when applied to an infinite lattice, needs some clarification. These questions were indeed the motivations of the present work.

For the sake of simplicity, we will here focus on the directed walk problem and sketch the derivation in this case. Various generalizations and detailed proofs will be given in a forthcoming paper. For the directed walk, the particle can only move to the right as a consequence of a very strong bias. Then the master equation simplifies ($W_n \equiv W_{n+1,n}$):

$$dp_n/dt = -W_n p_n(t) + W_{n-1} p_{n-1}(t) \tag{2}$$

Throughout this paper, two kinds of averages have to be taken in due time; in the following, overbarring represents an average with respect to the probabilities p_n, whereas brackets stand for an average on the disordered hopping rates W's, i.e., an average on all the configurations. Thus we set

$$\overline{x^m(t)} = \sum_{n=-\infty}^{\infty} n^m p_n(t) \tag{3}$$

$$\mu_q = \langle W_n^q \rangle \tag{4}$$

The moments μ_q are assumed to be bounded for $q < 0$; this physically means that there is no broken link. It is well known that only the negative ($q < 0$) moments are relevant for the dynamics at large times. We also define the Laplace transforms

$$p_n(z) = \int_0^{+\infty} p_n(t)\, e^{-zt}\, dt = L[p_n(t)] \tag{5}$$

Since each $p_n(t)$ is a positive bounded quantity, $p_n(z)$ is analytic everywhere in the right half-plane $\operatorname{Re} z > 0$. In addition, we set

$$x_m(z) = L[\overline{x^m(t)}] \tag{6}$$

With the initial condition $p_n(t=0) = \delta_{n,n_0}$, the master equation can be transformed into

$$zp_n(z) - \delta_{n,n_0} = -W_n p_n(z) + W_{n-1} p_{n-1}(z) \tag{7}$$

and has the unique solution

$$p_0(z) = 1/(z + W_0), \qquad p_{n>0}(z) = p_0(z) \prod_{j=1}^{n} W_{j-1}/(z + W_j) \tag{8}$$

whereas $p_{n<0}(z) = 0$, since the particle has no possibility to move to the left of its starting point. The formal expression for $x_m(z)$ can thus be written as

$$x_m(z) = \sum_{n=0}^{+\infty} (n^m/W_n) \prod_{j=0}^{n} u_j(Z) \tag{9}$$

where we have defined ($Z = z\mu_{-1}$)

$$u_j(Z) = (1 + Z/W_j\mu_{-1})^{-1} \qquad (j \geqslant 0) \tag{10}$$

We now use the master equation in order to obtain convenient expressions for $x_1(z)$ and $x_2(z)$; simple algebraic manipulations taking Eq. (7) into account show that

$$x_1(z) = z^{-1}S(Z, \xi = 1), \qquad x_2(z) = x_1(z) + (2/z)\{[\partial/\partial\xi]\, S(Z, \xi)\}_{\xi=1} \tag{11}$$

where the function $S(Z, \xi)$ is defined by

$$S(Z, \xi) = \sum_{n=0}^{+\infty} \xi^{n-1} \prod_{j=0}^{n} u_j(Z) \tag{12}$$

We aim at finding the large-time behavior of $\overline{x^m(t)}$; unfortunately, the series (12) is hopelessly useless as it stands. The clue for progressing towards this aim is in fact the possibility of resumming this last series through a procedure which we now briefly sketch. For each u_j we write

$$u_j(Z) = 1/(Z + 1 + Z\lambda_j) \tag{13}$$

where $\lambda_j = -1 + 1/\mu_{-1} W_j$ is a random number with zero mean; we now express u_j as the geometrical series

$$u_j(Z) = (1 + Z)^{-1} \sum_{n=0}^{+\infty} [-Z\lambda_j/(1 + Z)]^n \tag{14}$$

Then, by using this latter form in (12) and gathering terms involving the same λ_j, one obtains new geometrical series which can be readily summed up. Finally, $S(Z, \xi)$ turns out to have the convenient expansion

$$S(Z, \xi) = (1 - \xi + Z)^{-1} \Bigg[1 - Z \sum_{n=0}^{+\infty} \xi^{n-1}\lambda_n(1 + Z)^{-n} + Z^2 \sum_{n=0}^{+\infty} \xi^{n-1}(1 + Z)^{-(n+1)} \sum_{m=0}^{n} \lambda_n\lambda_m + \cdots \Bigg] \tag{15}$$

In this last equation, the dots denote terms which are at least cubic in the λ's and therefore involve moments μ_{-q} with $q > 2$. The basic idea underlying the above resummation is to move the center for an expansion of $x_m(z)$ from $z = 0$ (which contains all the information on the large-time regime but is a strong singularity) to a finite point on the negative axis ($z = -\mu_{-1}^{-1}$ i.e., $Z = -1$) which is less singular and contains the same information since transients have decayed and are invisible in the mean for

$t \gg \mu_{-1}$; the choice of this point for the expansion is motivated by the fact that the second term in the brackets of Eq. (15) has a vanishing disorder-average value (although it can have a very large fluctuation; see below). Note that all the λ's vanish identically in the ordered case. The expression (15) will play a central role by allowing for an easy analysis of the asymptotic dynamics. As will be detailed in a forthcoming paper, the same procedure allows us to deal with the general case [Eq. (1)] provided that one makes use of the auxiliary quantities $G_n^{\pm}(z)$ first introduced in ref. 3 (see also refs. 4 and 5), which are recursively constructed for a given sample $\{W_{n,m}\}$.

With the help of Eqs. (15) and (11), we readily get (in the directed case) for a given sample

$$x_1(z) = \mu_{-1}^{-1} z^{-2} + \cdots \tag{16}$$

where the first subdominant term has a zero mean and a mean square-root deviation diverging like $z^{-3/2}$ at small z. This implies that, for times large enough and for any given configuration of the W's, one has

$$\overline{x(t)} \sim Vt, \qquad V = 1/\mu_{-1} \tag{17}$$

This last equation displays two results, first, a finite velocity exists which coincides with the velocity found by Derrida, and, second, V does not depend on the given sample. In other words, a drift regime does exist at large times and is characterized by a nonfluctuating velocity. This last point can also be seen by calculating $\langle \overline{x(t)}^2 \rangle$ as a convolution integral (anyway required for the calculation of the mean-square deviation of the coordinate; see below) and by substracting $\mu_{-1}^{-2} t^2$; it can be seen that the relative difference is $\sim t^{-1/2}$.

In the general case [Eq.(1)], the same conclusions hold, formula (17) being replaced by

$$V = \langle 1/W_{n+1,n} \rangle \rangle^{-1} (1 - \langle W_{n,n+1}/W_{n+1,n} \rangle) \tag{18}$$

when $\langle W_{n,n+1} | W_{n+1,n} \rangle < 1$ (otherwise $V = 0$).

It is obvious, on physical grounds, that the waiting time for entering the regime described by Eq. (17) is at least equal to $W_<^{-1}$, where $W_<$ denotes the smallest hopping rate associated with a given configuration. This time is thus a strongly fluctuating quantity from one sample to another. A best estimate is provided by considering the first subdominant term for $x_1(z)$ [arising from the second term in the brackets of expression (15)] which converges in distribution (for any Z, Re $Z > 0$) toward a law with a mean square root deviation $\sim Z^{-1/2}$ in the limit $Z \to 0$. This implies that the first correction to $\overline{x(t)}$ has a *distribution over samples* with zero mean and a standard deviation *growing* like $t^{1/2}$ at large times.

On the other hand, it is easy to find the first correction to the average position

$$\langle \overline{x(t)} \rangle \sim V(t+t_0) \tag{19}$$

where t_0 is the time scale after which the strict drift regime is reached, on the average. In the directed case, one gets from (15)

$$t_0 = (\mu_{-2} - \mu_{-1}^2)/\mu_{-1} \tag{20}$$

This is a satisfactory result: the higher the disorder, the larger the time for entering the self-averaging drift regime (for an ordered lattice, this regime does exist at any time $t \geqslant 0$ and, on the other hand, it is well known that strong disorder, corresponding to diverging first moments, outside the class here considered, can lead to anomalous behaviors). It will appear below that the distance $d_0 = Vt_0 = (\mu_{-2} - \mu_{-1}^2)/\mu_{-1}^2$ can be viewed as a disorder-induced "dispersion length."

In the general case, making use of the $G_n^{\pm}$, t_0 is found to be

$$\begin{aligned} t_0 = {} & (1/V)(1 + \langle W_{n,n+1}/W_{n+1,n} \rangle)/(1 - \langle W_{n,n+1}/W_{n+1,n} \rangle) \\ & \times [V^2 \langle [1/G^+(0)]^2 \rangle - 1] \end{aligned} \tag{21}$$

with

$$\begin{aligned} \langle [1/G^+(0)]^2 \rangle = {} & [1 - \langle (W_{n,n+1}/W_{n+1,n})^2 \rangle]^{-1} \\ & \times [\langle 1/W_{n+1,n}^2 \rangle + (2/V) \langle W_{n,n+1}/W_{n+1,n}^2 \rangle] \end{aligned}$$

This expression holds provided $\langle (W_{n,n+1}/W_{n+1,n})^2 \rangle < 1$. If one would allow $\langle W_{n,n+1}/W_{n+1,n} \rangle < 1 < \langle (W_{n,n+1}/W_{n+1,n})^2 \rangle$, the characteristic time t_0 would diverge while the asymptotic velocity would still be non-zero.

The calculation of the diffusion constant is clearly much harder for a given sample than that of the velocity. In the general case, we were only able to obtain its average over disorder. This is still quite involved, since $x_2(z)$ is fairly complicated and since $\langle \overline{x(t)^2} \rangle$ implies the convolution $x_1 * x_1$. As expected, kinematical terms ($\sim t^2$) cancel each other and one finally gets

$$\langle \Delta_1 \overline{x^2(t)} \rangle = \langle \overline{x^2(t)} - (\overline{x(t)})^2 \rangle = 2Dt + O(t^{1/2}) \tag{22}$$

where D is the diffusion coefficient averaged over disorder. The $t^{1/2}$ correction has a pure (disorder-induced) statistical origin and, in the directed case, arises from modified Bessel functions. In this latter case, D has the expression

$$D = (1/2\mu_{-1})[1 + (\mu_{-2} - \mu_{-1}^2)/\mu_{-1}^2] = \mu_{-2}/2\mu_{-1}^3 \tag{23}$$

The first form of D displays the fact that, due to disorder, an additional spreading occurs which is revealed by the external bias, a result already found by Derrida. This additional spreading is related to the fact that, when the particle finds a small W, it becomes delayed as compared to the self-averaging drift motion induced by the bias field.

In the general case, one gets

$$D = (V/2)[(1 + \langle W_{n,n+1}/W_{n+1,n}\rangle)/(1 - \langle W_{n,n+1}/W_{n+1,n}\rangle) + Vt_0] \quad (24)$$

One recognizes, as in (23), besides the first term expected from the fluctuation-dissipation theorem, a disorder-induced additional spreading which precisely involves the time scale t_0 defined in (19). Using (21), one observes that the average over disorder of the single-particle diffusion constant (24) indeed coincides with Derrida's results for a periodized sample with $N \to +\infty$.

Since this is a demonstration about configuration averages, one would like to establish whether the diffusion constant is also a self-averaging quantity. We have been only able to prove that fluctuations do vanish for the directed walk [Eq. (2)]; even in this case, we only found a very tedious demonstration which requires space, care, and patience and for these reasons we here merely sketch it (explicit calculations will be given elsewhere).

In order to show that D does not fluctuate, we write, for a given sampling of the W's,

$$\Delta_1 \overline{x^2(t)} = 2D(\{W\})\, t + \cdots \quad (25)$$

where $D(\{W\})$ is the diffusion coefficient for this sample having the average value D given in Eq. (23). We now form the combination

$$\langle \Delta_2 \overline{x^2(t)} \rangle = \langle [\overline{x^2(t)} - (\overline{x(t)})^2]^2 \rangle - \langle \overline{x^2(t)} - (\overline{x(t)})^2 \rangle^2 \quad (26)$$

The second term in this last equation is known to behave as $4\langle D(\{W\})\rangle^2 t^2$ at large times [see Eq. (22)]; therefore, if it can be shown that the first one has the *same* time-dependence with the *same* coefficient, then one may conclude that

$$\langle [D(\{W\})]^2 \rangle = \langle D(\{W\}) \rangle^2 \quad (27)$$

which exhibits the fact that $D(\{W\})$ is indeed a nonrandom quantity and, as such, devoid of any fluctuation.

The explicit calculation of the first term in Eq. (26) is extremely tedious. Every quantity is represented by its Laplace transform, then averages over the disorder are taken. Numerous multiple inverse Laplace integrals [due to the nonlinearities present in expression (26)] then have to be calculated and analyzed in the vicinity of $z = 0$. Nevertheless, although very lengthy, the calculation does not give rise to serious difficulties. The result is

$$\langle \Delta_2 \overline{x^2(t)} \rangle / \langle \Delta_1 \overline{x^2(t)} \rangle^2 \sim O(t^{-1/2}) \tag{28}$$

where again the noninteger exponent originates from various statistical averages introducing Bessel functions. Equation (28) results from the exact cancellation of all the t^n $(2 \leqslant n \leqslant 4)$ terms and thus demonstrates the desired result anticipated in Eq. (27). Note that the time decay of the relative fluctuation is the same for D and V.

Let us now sum up our results:

(i) For an infinite disordered lattice and for the directed walk [Eq. (2)], the transport coefficients V and D are sample-to-sample independent (self-averaging) quantities. Although this is an expected result on physical grounds (on large space and time scales, the particle, provided it obeys a normal dynamics, moves enough to properly experience the disorder and average it), the explicit demonstration of this fact was lacking. Moreover, V and D are the same as those obtained for a periodized lattice in the limit of an infinite period.[1] Otherwise stated, the two limits $N \to +\infty$ and $t \to +\infty$ actually commute.

(ii) For the general hopping model [Eq. (1)], the same holds true for the velocity. We have shown that the disorder average of the diffusion constant D coincides with the result of ref. 1. It remains to be shown that, in this general case also, D is devoid of sample-to-sample fluctuations. This is indeed expected on physical grounds, since, most likely, a generalized form of the central limit theorem should hold for a given sample with a *limiting* form of the distribution of the position independent of the specific sample (and which is a standard Gaussian law when D and V both exist).

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